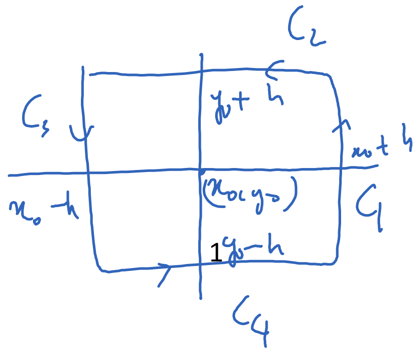
Lecture 23

Thursday, April 1, 2021 4:08 PM

* Prayer
* Spiritual thought
* Answering questions...

Green's theorem



$$C_1: \begin{cases} x = x_0+h \\ y = y_0+t \end{cases} \quad -h \le t \le h$$

$$C_2: \begin{cases} x = x_0-t \\ y = y_0+h \end{cases} \quad -h \le t \le h$$

$$C_3: \begin{cases} x = x_0-h \\ y = y_0-t \end{cases} \quad -h \le t \le h$$

$$C_4: \begin{cases} x = x_0+t \\ y = y_0-h \end{cases} \quad -h \le t \le h$$

$$\text{Circulation} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \int_{C_1} \underbrace{P\,dx}_{0} + Q\,dy + \int_{C_2} P\,dx + \underbrace{Q\,dy}_{0} + \int_{C_3} \underbrace{P\,dx}_{0} + Q\,dy + \int_{C_4} P\,dx + \underbrace{Q\,dy}_{0}$$

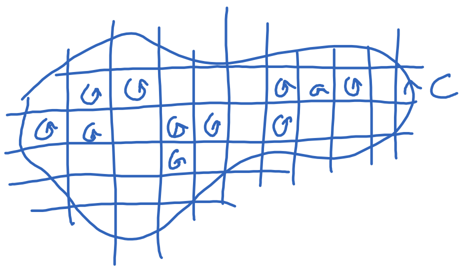
$$= \int_{C_1} Q\,dy + \int_{C_2} P\,dx + \int_{C_3} Q\,dy + \int_{C_4} P\,dx$$

$$= \int_{-h}^{h} \left[Q(x_0+h, y_0+t) - Q(x_0-h, y_0+t)\right] dt - \int_{-h}^{h} \left[P(x_0+t, y_0+h) - P(x_0-t, y_0+h)\right] dt$$

$$\approx 4h^2\, Q_x(x_0,y_0) - 4h^2\, P_y(x_0,y_0)$$

$$\text{Circulation density} = \frac{\text{total circulation}}{\text{area} = 4h^2}$$

$$= Q_x(x_0,y_0) - P_y(x_0,y_0).$$



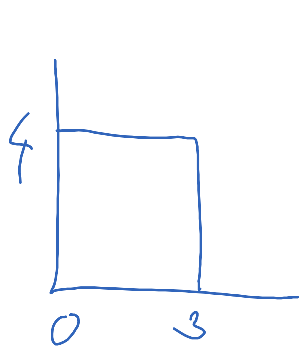
Total circulation

$$= \int_C F \cdot dr$$

$$\text{Total circulation} = \iint_D (Q_x - P_y)\, dA$$

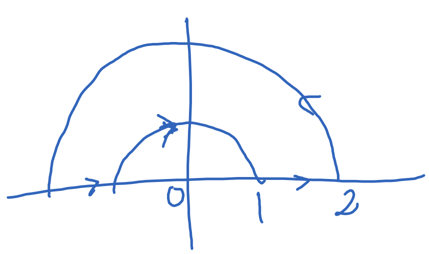
Green's theorem: $\boxed{\int_C F \cdot dr = \iint_D (Q_x - P_x)\, dA}$

Ex



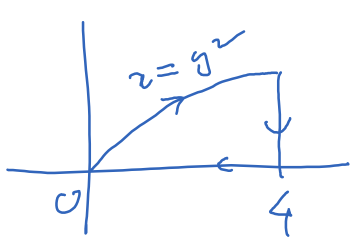
$$\int_C y e^x dx + 2e^x dy$$ ← already given last time

Ex



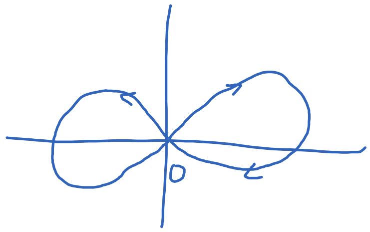
$$\int_C y^2 dx + 3xy\, dy$$

Ex



$$\int_C (x^{2/3} + y^2)\, dx + (y^{4/3} - x^2)\, dy$$

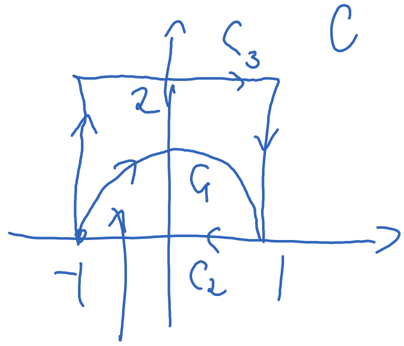
Ex



$$C: \begin{cases} x = \sin t \\ y = \sin t \cos t \end{cases} \qquad 0 \le t \le 2\pi$$

$$\int_C x\,dx + y\,dy$$

Ex

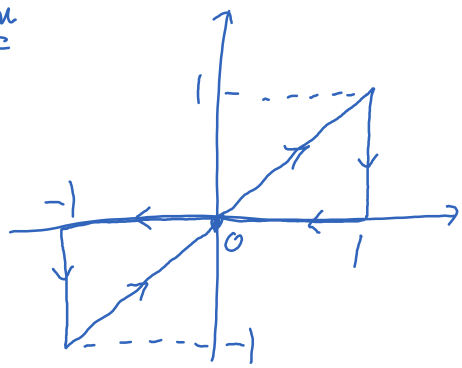


$$C = C_1 + C_2 + C_3 + C_2$$
$$= (C_1 + C_2) + (C_3 + C_2)$$

$y = 1 - x^2$

$$\int_C x\,dx + y\,dy = ?$$

Ex



$$\int_C F \cdot dr = ?$$

$$F(x,y) = \langle x+y,\ x-y \rangle$$

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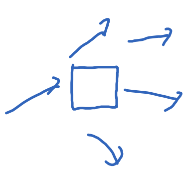
* Curl and divergence



divergence = flux density = $\nabla \cdot F$

curl is a vector = $\nabla \times F$

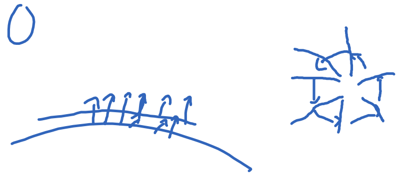
gradient is a vector = ∇f

$$\text{curl } F = \nabla \times F = \det \begin{pmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= \left\langle \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_3} \right\rangle$$

$$\text{div } F = \nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

$$\text{curl}(\nabla f) = 0$$



macroscopic rotation $\neq$ microscopic rotation

$F(x,y) = \langle -y, x, 0 \rangle$ has curl $= 2$.

$F(x,y) = \langle y, 0, 0 \rangle$ doesn't seem to rotate in macroscopic scale, but not microscopic scale.

$F(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$ doesn't seem to rotate in microscopic scale, but not in macroscopic scale.

$$\boxed{\text{div}(\text{curl } F) = 0}$$

<u>Ex</u> curl $G = \langle x, y, z \rangle$

what is G?